P: ISSN NO.: 2394-0344 RNI No.UPBIL/2016/67980 VOL-2* ISSUE-6* September- 2017
E: ISSN NO.: 2455-0817 Remarking An Analisation

Softly Z* Normal Spaces

Abstract

The aim of this paper is to introduce a new class of softly normal called soft Z*-normality by using Z*-open sets and obtained several properties of such a space. Moreover, we obtain some new characterizations and preservation theorems of soft Z*-normality.

Nidhi Sharma
Assistant Professor,
Deptt.of Mathematics,
N.I.E.T.
Greater Noida

Neeraj Kumar Tomar
Assistant Professor,
Deptt.of Mathematics,
N.R.E.C. College,
Khurja

Keywords: π-closed, Z*-closed, α-closed sets, softly Z*-normal spaces.

Introduction

In this paper, we introduced the new concept of softly Z*-normal by using Z*-open set due to Ali Mubarki [1] and obtained several properties of such a space. Recently, M. C. Sharma and Hamant Kumar [4] introduced a weaker version of normality called softly-normality and prove that soft-normality is a property, which is implied by quasi-normality and almost-normality and obtained several properties of such space. We prove that soft Z *-normality is a topological property and it is a hereditary property with respect to closed domain subspace. Moreover, we obtain some new characterizations and preservation theorems of softly Z*-normal spaces. Throughout this paper, (X, τ), (Y, σ) spaces always mean topological spaces X, Y respectively on which no separation axioms are assumed unless explicitly stated.

2010 AMS Subject Classification

54D15

Preliminaries

Definition

A subset A of a topological space X is called,

1. α-closed [3] if $cl(int(cl(A))) \subseteq A$.
2. Z*-closed [1] if $int(cl(A)) \cap cl(\delta\text{-}int(A)) \subseteq A$.
3. Regular closed [10]) if A = cl(int(A)).

The complement of α-closed (resp. Z*-closed, regular closed) set is called α-open (resp. Z*-open, regular open) set. The intersection of all Z*-closed sets containing A is called the Z*-closure of A and denoted Z*-cl(A).The union of all Z*-open subsets of X which are contained in A is called the Z*-interior of A and denoted by Z*-int(A). The finite union of regular open sets is said to be □-open. The complement of a □-open set is said to be □-closed.

Definitions stated in preliminaries, we have the following diagram:

closed ⇒ α-closed ⇒ Z*-closed

However the converses of the above are not true may be seen by the following examples.

Example

Let X = {a, b, c, d } and τ = { ϕ, {a}, {b}, {a, b},{a, b, c}, X}.Then the set A = {c} is α-closed set as well as Z*-closed set but not closed set in X .

Remark

Every regular open (resp. regular closed) set is π-open (resp. π-closed).

Softly Z*- Normalspaces

Definition

A topological space X is said to be Softly normal [4](softly Z*-normal) if for any two disjoint closed subsets A and B of X, one of which is π-closed and other is regularly closed, there exist disjoint open(Z*-open) sets U and V of X such that $A \subset U$ and $B \subset V$.

Almost-normal [7] (**almost Z*-normal [5]**)

If for every pair of disjoint sets A and B, one of which closed and other is regularly closed, there exist disjoint open (Z*- open) sets U and V of X such that $A \subset U$ and $B \subset V$.

Quasi Normal [**11**] (**quasi Z*-Normal [6]**)

If for any two disjoint π-closed subsets A and B of X, there exist disjoint open (Z*-open) sets U and V of X such that A⊂ U and B⊂ V.

π-Normal [**2**]

If for any two disjoint closed subsets A and B of X, one of which is π-closed, there exist disjoint open sets U and V of X such that A ⊂ U and B ⊂ V.

Mildly Normal [**8,9**]

If for any two disjoint regularly closed subsets A and B of X, there exist disjoint open sets U and V of X such that A ⊂ U and B ⊂ V.

By the definitions stated above, we have the following diagrams:

Quasi-normal ⇒ quasi Z*-normal ⇒ soft Z*-normal ⇒ mildZ*-normal
⇑
normal ⇒ π-normal ⇒ almost normal ⇒ softly normal ⇒ mildly normal
⇓ ⇓ ⇓
⇓ ⇓
Z*-normal ⇒ πZ*-normal ⇒ almost Z*-normal ⇒ soft Z*-normal⇒mild Z*-normal

Where none of the implications is reversible as can be seen from the following examples:

Example

Let X = {a, b, c, d} and τ = {ϕ, {a}, {c}, {a, c},{ b, d}, {a, b, d}, {b, c, d}, X }. The pair of disjoint π-closed subsets of X are A = {a} and B = {c}. Also U = {a} and V = {b, c, d} are disjoint open sets such that A ⊂ U and B ⊂ V. Hence X is quasi-normal as well as quasi Z*-normal as well as softly Z*-normal because every open set is Z*-open set.

Example

Let X = {a, b, c, d} and τ = {ϕ, {a}, {b}, {a, b}, {c, d}, {a, c, d}, {b, c, d}, X}. Then A = {b} is closed and B = {a} is regularly closed sets there exist disjoint open sets U = {b, c, d} and V = { a} of X such that A ⊂ U and B ⊂ V. Hence X is almost normal as well as almost Z*-normal as well as softly Z*-normal because every open set is Z*-open set.

Example

Let X = {a, b, c, d} and τ = {ϕ, {b, d}, {a, b, d}, {b, c, d}, X}. The pair of disjoint closed subsets of X are A = {a} and B = {c}. Also U = {a, b} and V = {c, d} are Z*-open sets such that A ⊂ U and B ⊂ V. Hence X is Z*-normal but it is not normal.

Example

Let X = {a, b, c} and τ = {ϕ, {a}, {a, b}, {a, c}, X}. Then (X, τ) is almost -normal as well as almost Z*-normal, but it is not Z*-normal, since the pair of disjoint closed sets {b} and {c} have no disjoint Z*-open sets containing them. But it is not normal.

Example

Let X = {a, b, c, d} and τ = {ϕ, {a}, {b}, {d}, {a, b}, {a, d}, {b, d}, {a, b, c}, {a, b, d}, X}. Then X is Z*-normal.

Theorem

For a topological space X, the following are equivalent:

a. X is softly Z*-normal.
b. For every π-closed set A and every regularly open set B with A ⊂ B, there
c. exists a Z*-open set U such that A ⊂ U ⊂ Z*-cl(U) ⊂ B.

For every regularly closed set A and every π-open set B with A ⊂ B, there exists a Z*-open set U such that A ⊂ U ⊂ Z*-cl(U) ⊂ B.

d. For every pair consisting of disjoint sets A and B, one of which is π-closed and the other is regularly closed, there exist Z*-open sets U and V such that A ⊂ U, B ⊂ V and Z*-cl(U) ∩ Z*-cl(V) = ϕ

Proof

(a) ⇒(b). Assume (a). Let A be any π-closed set and B be any regularly open set such that A ⊂ B. Then A ∩ (X – B) = ϕ where (X – B) is regularly closed. Then there exist disjoint Z*-open sets U and V such that A ⊂ U and (X – B) ⊂ V. Since U ∩ V= ϕ, then Z*-cl(U) ∩ V = ϕ. Thus Z*-cl(U) ⊂ (X – V) ⊂ (X – (X – B)) = B. Therefore, A ⊂ U ⊂ Z*-cl(U) ⊂ B.

(b) ⇒ (c). Assume (b). Let A be any regularly closed set and B be any π-open set such that A ⊂ B. Then, (X – B) ⊂ (X – A), where (X – B) is π-closed and (X – A) is regularly open. Thus by (b), there exists a Z*-open set W such that (X – B) ⊂ W ⊂ Z*-cl(W) ⊂ (X – A). Thus A ⊂ (X – Z*-cl(W)) ⊂ (X – W) ⊂ B. So, we let U = (X – Z*-cl(W)), which is Z*-open and since W ⊂ Z*-cl(W), then (X – Z*-cl(W) ⊂ (X – W). Thus U ⊂ (X – W), hence Z*-cl(U) ⊂ Z*-cl(X – W) = (X – W) ⊂ B.

(c) ⇒ (d). Assume (c). Let A be any regular closed set and B be any π-closed set with A ∩ B = ϕ. Then A ⊂ (X – B), where (X – B) is π-open. By (c), there exists a Z*-open set U such that A ⊂ U ⊂ Z*-cl(U) ⊂ (X – B). Now, Z*-cl(U) is Z*-closed. Applying (c) again we get a Z*-open set W such that A ⊂ U ⊂ Z*-cl(U) ⊂ W ⊂ Z*-cl(W) ⊂ (X – B). Let V = (X – Z*-cl(W)), then V is Z*-open set and B ⊂ V. We have (X – Z*-cl(W)) ⊂ (X – W), hence V ⊂ (X – W), thus Z*-cl(V) ⊂ Z*-cl(X – W) = (X – W). So, we have Z*-cl(U) ⊂ W and Z*-cl(V) ⊂ (X – W). Therefore Z*-cl(U) ∩ Z*-cl(V) = ϕ.

(d) ⟹ (a) is clear.

Theorem

For a topological space X, the following are equivalent:

a. X is softly Z*-normal.
b. For every pair of sets U and V, one of which is π-open and the other is regular open whose union is X, there exist Z*-closed sets G and H such that G ⊂ U, H ⊂ V and G ∪ H = X.
c. For every π-closed set A and every regular open set B containing A, there is a Z*-open set V such that A ⊂ V ⊂ Z*-cl(V) ⊂ B.

Proof

(a) ⇒ (b). Let U be a π-open set and V be a regular open set in a softly Z*-normal space X such that U ∪ V = X. Then (X – U) is π-closed set and (X – V) is regular closed set with (X – U) ∩ (X – V) = ϕ. By soft Z*-normality of X, there exist disjoint Z*-open sets U_1 and V_1 such that X – U ⊂ U_1 and X – V ⊂ V_1. Let G = X – U_1 and H = X – V_1. Then G and H are Z*-closed sets such that G ⊂ U, H ⊂ V and G ∪ H = X.

(b) ⇒ (c) and (c) ⇒ (a) are obvious.

Using Theorem 3.7, it is easy to show the following theorem, which is a Urysohn's Lemma version for soft Z*-normality. A proof can be established by a similar way of the normal case.

Theorem

A space X is softly Z*-normal if and only if for every pair of disjoint closed sets A and B, one of which is π-closed and other is regularly closed, there exists a continuous function f on X into [0, 1], with its usual topology, such that f(A) = {0} and f(B) = {1}.

It is easy to see that the inverse image of a regularly closed set under an open continuous function is regularly closed and the inverse image of a π-closed set under an open continuous function π-closed. We will use that in the next theorem.

Theorem

Let X is a softly Z*-normal space and f : X → Y is an open continuous injective function. Then f(X) is a softly Z*-normal space.

Proof

Let A be any π-closed subset in f(X) and let B be any regularly closed subset in f(X) such that A ∩ B = ϕ. Then $f^{-1}(A)$ is a π-closed set in X, which is disjoint from the regularly closed set $f^{-1}(B)$. Since X is softly Z*-normal, there are two disjoint open sets U and V such that $f^{-1}(A)$ ⊂ U and $f^{-1}(B)$ ⊂ V. Since f is one-one and open, result follows.

Corollary

Soft Z*-normality is a topological property.

Lemma

Let M be a closed domain subspace of a space X. If A is a Z*-open set in X, then A ∩ M is Z*-open set in M.

Theorem

A closed domain subspace of a softly Z*-normal is softly Z*-normal.

Proof

Let M be a closed domain subspace of a softly Z*-normal space X. Let A and B be any disjoint closed sets in M such that A is regularly closed and B is π-closed. Then, A and B are disjoint closed sets in X such that A is regularly closed and B is π-closed in X. By soft Z*-normality of X, there exist disjoint Z*-open sets U and V of X such that A ⊂ U and B ⊂ V. By the Lemma 3.12, we have U ∩ M and V ∩ M are disjoint Z*-open sets in M such that A ⊂ U ∩ M and B ⊂ V ∩ M. Hence, M is softly Z*-normal subspace.

Since every closed and open (clopen) subset is a closed domain, then we have the following corollary.

Corollary

Soft Z*-normality is a hereditary with respect to clopen subspaces.

Conclusion

In this paper, we have introduced weak form of normal space namely soft Z*-normality and established their relationships with some weak forms of normal spaces in topological spaces.

References

1. *Ali Mubarki, Z*-open sets and Z*-continuity in topological spaces, Internat. J. of Math. Archive, 3(1), 2012, 118 -125.*
2. *L. N. Kalantan, π-normal toplogical spaces, Filomat 22:1 (2008), 173-181.*
3. *O. Njastad, On some classes of nearly open sets, Pacific J. Math., 15(1965), 961- 970.*
4. *M.C.Sharma and Hamant Kumar, Softly normal topological spaces accepted for publication in the Journal of Acta Ciencia Indica.*
5. *M. C. Sharma and Poonam Sharma, Almost Z*-normal spaces, J. Math. Sci. Vol.7(4), 2012,343-350.*
6. *M. C. Sharma and Poonam Sharma, Quasi Z*-normal spaces, Acta Ciencia Indica , XXXIX M 2013 No.4, 437- 442.*
7. *M. K. Singal and S. P. Arya, Almost normal and almost completely regular spaces, Glasnik Mathematicki Tom 5(25) 1(1970), 141-152*
8. *M. K. Singal and A. R. Singal, Mildly normal spaces, Kyungpook Math. J. ,13(1973), 27-31.*
9. *E. V. Shchepin, Real functions and near normal spaces, Sibirskii Mat. Zhurnal, 13(1972), 1182-1196.*
10. *M. Stone, Application of the theory of Boolean rings to general topology, Trans. Amer. Math.Soc. 41(1937), 374 - 481.*
11. *V. Zaitsev,On certain classes of topological spaces and their biocompactifications, Dokl. Akad. Nauk SSSR, 178(1968), 778 -779.*